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LETTER TO THE EDITOR

The missing link: operators for labelling multiplicity in the Clebsch–Gordan series

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Abstract. It is indicated that there are invariant operators on the direct product space of irreducible representations of Lie groups which distinguish between repeating representations. They complete the labelling of the Clebsch-Gordan series.

Consider decomposition into irreducible representations (IR) (de Swart 1963) of the direct product of two octet representations of SU(3):

$$\underline{8} \otimes \underline{8} = \underline{1} \oplus \underline{8}_S \oplus \underline{8}_A \oplus \underline{10} + \underline{10}^* + \underline{27}. \tag{1}$$

There are two octet representations on the right-hand side. They have identical Young's tableau and therefore identical values of Casimir invariants. Such a multiplicity of IRs is quite common in the Clebsch-Gordon series. It is often stated in the literature that one has to go outside the group to find the operator to distinguish such IRs. Generally one uses symmetry properties of the wavefunction for this purpose. For instance, in the example (1), the two octets are respectively symmetric and antisymmetric under the interchange of the representations on the left-hand side. The labels S and A represent this feature. Even if one accepts all this, there is no systematic specification of the operator(s) T which formally distinguishes the degeneracy of the IRs.

This is an unsatisfactory situation and has blocked progress in the Clebsch-Gordan theory for groups other than SU(2).

In case of SU(2), a given IR appears at most once in the direct product of two IRs. This allows the states of the decomposition to be uniquely labelled by $|j_1 j_2 jm\rangle$ where j_1, j_2 and j simply label the angular momenta of the two initial IRs and of the sum. In addition, we are in the fortunate situation that the values that j can take for given j_1 and j_2 are given by a simple algorithm—the triangle rule.

To put the situation for other groups in the right perspective, we first consider the decomposition into IRs of a direct product of three IRs of SU(2). It is well known that in this case a given IR may appear more than once. For example,

$$\underline{2} \otimes \underline{2} \otimes \underline{2} = \underline{2} \oplus \underline{2} \oplus \underline{4}. \tag{2}$$

However, labelling the series is completely satisfactory in this case. We denote the three angular momenta by $\boldsymbol{J}_1$, $\boldsymbol{J}_2$ and $\boldsymbol{J}_3$. We have first to 'add' any two of the three. If, for instance, $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are added, the resulting representations are uniquely labelled by the operators $\boldsymbol{J}_1^2$, $\boldsymbol{J}_2^2$ and $\boldsymbol{J}_{12}^2$ where $\boldsymbol{J}_{12} = \boldsymbol{J}_1 + \boldsymbol{J}_2$. Next, we have to add $\boldsymbol{J}_3$ to $\boldsymbol{J}_{12}$.

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The resulting representations are uniquely labelled by J_1^2, J_2^2, J_3^2, J_{12}^2, J^2 where $J = J_1 + J_2 + J_3$. We now see that we need one additional label J_{12}^2 in addition to the Casimirs J_1^2, J_2^2, J_3^2 of the initial IRs and J^2, the Casimir of the final IR. The operator J_{12}^2 serves to distinguish final states of identical total angular momenta. We also know that J_{12}^2 is enough to make the distinction.

We now indicate that the situation for other groups is not very different. The change is that what happens in case of SU(2) for the addition of three angular momenta, already occurs for the direct product of *two* IRs in the case of the other groups.

Consider the Lie algebra for SU(N). The generators are conveniently denoted by X_j^i with $(X_j^i)_3^+ = X_i^j$ and $\Sigma_i X_j^i = 0$, $i, j, = 1, 2, \ldots, N$. The Lie algebra is

$$[X_j^i, X_l^k] = \delta_l^i X_j^k - \delta_j^k X_l^i \tag{3}$$

where δ_j^i is the Kronecker delta. The IRs may be labelled by the Casimir–Racah operators $I_n(X) \equiv \mathrm{tr}(X^n)$, $n = 2, 3, 4, \ldots$ and

$$I_1(X) \equiv \text{`det } X\text{'} \equiv \frac{1}{N!} \varepsilon_{i_1 \ldots i_n} \varepsilon_{j_1 \ldots j_n} X_{j_1}^{i_1} X_{j_2}^{i_2} \ldots X_{j_n}^{i_n} \tag{4}$$

where $\varepsilon_{i_1 i_2 \ldots i_n}$ is the totally antisymmetric tensor in N indices. Each of these operators commute with every element of the Lie lagebra and $(N-1)$ of them are 'algebraically independent'.

Consider direct product of two IRs of SU(N). The generators on the respective representation spaces are denoted by X_j^i and Y_j^i, $i, j = 1, N$. These IRs are respectively labelled by the eigenvalues of the (algebraically independent set from) $I_n(X)$ and $I_n(Y)$, $n = 1, 2, \ldots$. The elements in the decomposition are usually labelled by $I_n(X + Y)$, $n = 1, 2, \ldots$, the invariants of the sum of the generators. But these have same eigenvalue for a given IR of SU(N) and therefore do not distinguish between repeating IRs in the Clebsch–Gordan series.

However one can construct other invariant operators on the direct product space such as

$$I_{m_1 n_1 m_2 n_2 \ldots}(X, Y) \equiv \mathrm{tr}(X^{m_1} Y^{n_1} X^{m_2} Y^{n_2} \ldots)$$

(where $m_1, n_1, m_2, n_2 \ldots = 1, 2, \ldots$) or the analogue of 'det X', equation (4) with some Xs on the right-hand side replaced by Ys. The Casimir–Racah invariants of each of the initial IRs commute with any such operator because they commute with any X_j^i and Y_j^i, $i, j = 1, N$. It is also easy to argue that these new invariants commute with any element of $(X + Y)_j^i$ and therefore with $I_n(X + Y)$, Casimir–Racah invariants of the sum. For this consider a finite element, $U = \exp(\mathrm{i}\theta_j^i (X + Y)_i^j)$ where $\theta_j^i = -\theta_i^j$, $\Sigma_i \theta_j^i = 0$, $i, j = 1, N$ parametrizes the element. U makes an equal SU(N) rotation on X and Y:

$$UX_j^i U^\dagger = (uXu^\dagger)_j^i \qquad UY_j^i U^\dagger = (uYu^\dagger)_j^i. \tag{5}$$

Here $(uXu^\dagger)_j^i \equiv u_k^i X_l^k u_j^{\dagger l}$ and $u = \exp(\mathrm{i}\theta_j^i T_i^j)$ where T_j^i are the generators of SU(N) in the defining representation. Using that u is a SU(N) matrix, we see that the new invariants commute with U and hence with each $(X + Y)_j^i$, $i, j = 1, N$.

This means that we may use these new invariants to label the Clebsch–Gordan series in addition to the usual Casimir–Racah invariants of X, Y and $X + Y$. In general, these new invariants have distinct eigenvalues for the repeating IRs and therefore serve to label the multiplicity. It is to be noted that, though the new invariants commute with the Casimir–Racah invariants of X, of Y and of $(X + Y)$, and also with each $(X + Y)_j^i$, they do not commute with each other, in general. One has to choose the maximal commuting set from among them, to label the series. Moreover, just as with $I_n(X)$, the new invariants are not all algebraically independent. However we do not address such general questions here. We only illustrate our contentions in the context of example (1).

$\operatorname{tr} X^2$ and $\operatorname{tr} X^3$ are generally used to label the IRs of SU(3). (This set does not distinguish between conjugate representations and is therefore, strictly speaking, not the correct choice. 'det x' in place of $\operatorname{tr} X^3$ removes this deficiency. However, for simplifying our illustration, we continue to use $\operatorname{tr} X^3$. Our analysis applies equally well for det X.) The usual labels for the Clebsch–Gordan series are $\operatorname{tr} X^2$, $\operatorname{tr} X^3$, $\operatorname{tr} Y^2$, $\operatorname{tr} Y^3$, $\operatorname{tr}(X+Y)^2$ and $\operatorname{tr}(X+Y)^3$. The last two may be equivalently replaced by $\operatorname{tr} XY$ and $\operatorname{tr} X^2Y+\operatorname{tr} XY^2$, involving a linear combination of the set. This set does not distinguish between multiplicites. As an example of the new invariants we may choose a linear combination of $\operatorname{tr} X^2Y$ and $\operatorname{tr} XY^2$, a combination linearly independent of $\operatorname{tr} X^2Y+\operatorname{tr} XY^2$ used above. Examples are: $\operatorname{tr} X^2Y$, $\operatorname{tr} XY^2$ and $\operatorname{tr} X^2Y-\operatorname{tr} XY^2$.

We now compute the eigenvalues of $\operatorname{tr} X^2Y$ for $\underline{8}_S$ and $\underline{8}_A$ in (1) and show that this operator serves to distinguish between the two octets. We denote the basis for an octet of SU(3) by $|H^i_j\rangle$ (following the quark model) where $|H^i_j\rangle \equiv -|H^j_i\rangle$, $\Sigma_i\,|H^i_i\rangle=0$, $i, j=1, 2$ or 3. The basis for the two octets of the initial representation in (1) is denoted by $|B^i_j\rangle$ and $|M^i_j\rangle$ respectively. The transformation law for the octet is

$$X^i_j|B^k_l\rangle=|-(T^i_jB)^k_l+(BT^i_j)^k_l\rangle$$

where the right-hand side stands for

$$-(T^i_j)^k_m|B^m_l\rangle+(T^i_j)^m_l|B^k_m\rangle.$$

The matrices T^i_j have the matrix elements, $(T^i_j)^k_l\equiv\delta^i_l\delta^k_j-\frac{1}{3}\delta^i_j\delta^k_l$. The action of Y^i_j on $|H^k_l\rangle$ is given by a similar formula. We now have,

$$\begin{aligned}
&\operatorname{tr}(X^2Y)(|B^i_j\rangle\otimes|M^k_l\rangle)\\
&\quad=X^p_q(|-(T^q_rB)^i_j+(BT^q_r)^i_j\rangle\otimes|-(T^r_pM)^k_l+(MT^r_p)^k_l\rangle)\\
&\quad=(|(-T^q_r(-T^p_qB+BT^p_q))^i_j+((-T^p_qB+BT^p_q)T^q_r)^i_j\rangle)\\
&\qquad\otimes|-(T^r_pM)^k_l+(MT^r_p)^k_l\rangle\\
&\quad=|3B^k_jM^i_l+3\delta^i_l(MB)^k_j+\delta^i_j(BM-MB)^k_l\rangle.
\end{aligned}$$

The two octets $|8_S\rangle$ and $|8_A\rangle$ are given by

$$|8_S\rangle=|\{B, M\}^i_j-\tfrac{2}{3}\delta^i_j\operatorname{tr} BM\rangle$$

$$|8_A\rangle=|[B, M]^i_j\rangle$$

the (traceless parts) of the anti-commutator and commutator respectively. We get

$$\operatorname{tr}(X^2Y)|8_S\rangle=9|8_S\rangle-5|8_A\rangle$$

$$\operatorname{tr}(X^2Y)|8_A\rangle=-9|8_S\rangle+9|8_A\rangle.$$

This means $\operatorname{tr}(X^2Y)$ selects our certain linear combinations of $|8_S\rangle$ and $|8_A\rangle$ and assigns them distinct eigenvalues. Therefore $\operatorname{tr}(X^2Y)$ distinguishes the two octets.

We find that no linear combination of $\operatorname{tr} X^2Y$ and $\operatorname{tr} XY^2$ gives $|8_S\rangle$ and $|8_A\rangle$ (used in the quark model) as the eigenvectors with distinct eigenvalues. Perhaps we need invariants of higher degree in X and Y for this purpose.

In this letter we have demonstrated that there are invariant operators in the direct product space of IRs which distinguish between repeating representations in the Clebsch–Gordon series. The problem of constructing a minimal set of algebraically independent invariant operators which completely label the Clebsch–Gordan series for a general group will be addressed elsewhere.

Reference

de Swart J J 1963 *Rev. Mod. Phys.* **35** 916